ON COMMUTATIVE WEAK BCK-ALGEBRAS

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Abstract. The class of weak BCK-algebras is obtained by replacing (in the standard axiom set by K. Iseki and S. Tanaka) the first BCK axiom $(x - y) - (x - z) \leq x - z$ by its weakening $x \leq z \Rightarrow x - y \leq x - z$. It is known that every weak BCK-algebra is completely determined by the structure of its initial segments. We review several classes of commutative weak BCK-algebras, show that they are varieties, and characterize the initial segments of algebras in each of these classes as lattices equipped with a suitable kind of complementation. In particular, commutative weak BCK-algebras are just those meet semilattices with the least element in which all initial segments are non-distributive de Morgan lattices.

1. Introduction

Like BCK-algebras presented in [28, 29], a weak BCK-algebra is a poset with the bottom element and equipped with a binary operation considered as subtraction. Both BCK-algebras and weak BCK-algebras have also been treated in the dual form as algebras with the reversed ordering, the top element and an operation considered as implication. We hold here the former viewpoint, and call algebras presented in the latter form BCK*-algebras, resp., weak BCK*-algebras. The class of weak BCK*-algebras was introduced by the author in the talk on the 44-th Summer School on General Algebra and Ordered Sets in Radějov (Czech Republic), September 2006. They were studied more extensively in [23] and latter, already as algebras with subtraction, in [24, 26]. A representation theorem for a subclass of weak BCK*-algebras was proved in [25].

Definition 1.1. A *weak BCK-algebra* (*wBCK-algebra*, for short) is an algebra $(A, -, 0)$, where A is a poset with 0 the least element, and $-$ is a total binary operation on A (called *subtraction*) satisfying the axioms

$(-_1)$: $x \leq y$ if and only if $x - y = 0$,
$(-_2)$: if $x - y \leq z$, then $x - z \leq y$.

For instance, every poset with 0 carries a wBCK-algebra, where

$$x - y = 0 \text{ if } x \leq y, \text{ and } x - y = x \text{ otherwise.}$$

Such a wBCK-algebra will be called *discrete*.

The operation $-$ induces on every initial segment (or section) $[0, p]$ of a wBCK-algebra A a unary operation ${}^+_p$ defined by $x^+_p := p - x$, which can be considered as a kind of complementation on $[0, p]$, named in [24] a g*-complementation. It is

2010 *Mathematics Subject Classification.* Primary 06F35; Secondary 03G25, 06B75, 06C15.

Key words and phrases. commutative, De Morgan complementation, Galois complementation, implicative, nearlattice, orthocomplementation, orthoimplicative, positive implicative, quasi-BCK-algebra, sectionally orthocomplemented, sectionally semicomplemented, weak BCK-algebra.

This work was supported by ESF project No. 2009/0216/1DP1.1.1.2.0/09/APIA/VIAA /044.

shown there that every wBCK-algebra is completely determined by the structure of its sections; even more, there is bijective connection between wBCK-algebras and certain sectionally g*-complemented posets with zero (see below Section 6 for details). Also, connections between several classes of wBCK-algebras and properties of respective sectional g*-complementations were found out in [24] .

In the present paper we deal mainly with the narrower class of commutative wBCK-algebras, i.e., those fulfilling the identity $x - (x - y) = y - (y - x)$. Sections 2 and 3 review some basic facts concerning general wBCK-algebras and, respectively, commutative wBCK-algebras. In particular, like BCK-algebras, every commutative wBCK-algebra is a meet semilattice, and all these algebras form a variety. In Section 4, isolated is its subclass of so called orthoimplicative wBCK-algebras, which satisfy the condition $x - y \le y \Rightarrow x \le y$, This class also is a variety. Further, in Section 5, orthoimplicative wBCK-algebras satisfying the condition $x \le y \le z \Rightarrow z - x \le z - y$, called implicative wBCK-algebras, are shown to form another variety. At last, in Section 6, we characterize the sectional g*-complementations $\overset{+}{_p}$ in commutative, orthoimplicative and implicative wBCK-algebras. In particular, initial sections in such algebras are non-distributive De Morgan lattices, ortholattices and orthomodular lattices, respectively. Recall that in implicative BCK-algebras the sections are Boolean lattices.

We also disclose that many algebras with implications known in the literature are, in fact, commutative wBCK*-algebras of that or other type.

2. Weak BCK-algebras

2.1. **Preliminaries.** To make the structure of formulas more transparent, we follow [24, 27] and use dots instead of most parentheses. For instance, both expressions

$$x - y. - .z - x\colon - .(x - \colon y - .z - x) - y,$$
$$(x - y. - .z - x) - (x - \colon y - .z - x.\therefore - y)$$

are condensed versions of $((x - y) - (z - x)) - ((x - (y - (z - x))) - y)$.

The subsequent list of basic properties of subtraction is borrowed from [23, 24].

Proposition 2.1. *In any wBCK-algebra,*

$(-_3)$: $x - .x - y \le y$,
$(-_4)$: *if* $x \le y$, *then* $z - y \le z - x$,
$(-_5)$: $x - \colon x - .x - y = x - y$,
$(-_6)$: $x - x = 0$,
$(-_7)$: $x - y \le x$,
$(-_8)$: $x - .x - y \le x$,
$(-_9)$: $x - 0 = x$,
$(-_{10})$: $0 - x = 0$.

It is easily seen that the axiom $(-_2)$ can be derived from $(-_3)$ and $(-_4)$. Moreover, the axiom $(-_1)$ can be derived using $(-_2)$ and $(-_9)$: $x - y \le 0$ iff $x - 0 \le y$ iff $x \le y$. These observations allow us to simplify the axioms of wBCK-algebras.

Proposition 2.2 ([24]). *A poset with a least element* 0 *and a binary operation* $-$ *is a weak BCK-algebra if and only if* $(-_3)$, $(-_4)$ and $(-_9)$ *are fulfilled in it.*

We have included the order relation $\le$ among the primitives to get a more concise axiom set. Of course, the equivalence $(-_1)$ may be considered as a definition of the

relation in terms of subtraction and 0, and then the assumption that A is a poset can be replaced by explicit order axioms $(-_{10})$, $(-_6)$ and

$$\text{if } x - y = 0 \text{ and } y - x = 0, \text{ then } x = y. \tag{2.1}$$

The transitivity axiom,

$$\text{if } x - y = 0 \text{ and } y - z = 0, \text{ then } x - z = 0,$$

is derivable in thus reorganized axiom system by a use of $(-_4)$, $(-_{10})$ and (2.1): if $y - z = 0$, then $x - z. - .x - y = 0$, and if, in addition, $x - y = 0$, then $x - z. - 0 = 0 = 0 - .x - z$. So, $x - z = 0$.

Corollary 2.3. *An algebra $(A, -, 0)$ is a wBCK-algebra with respect to the relation $\leq$ defined on it by $(-_1)$ if and only if it satisfies $(-_3)$, $(-_4)$, $(-_6)$, $(-_{10})$ and (2.1).*

We note without proof that, due to $(-_3)$, the axioms $(-_6)$ and $(-_{10})$ here may be replaced by one identity

$(-_{11})$: $x - .0 - y = x$.

BCK-algebras are defined in [28, 29] as algebras $(A, 0, -)$ satisfying the same axioms (listed in the corollary) with the exception of $(-_4)$, which actually replaces here the (stronger) BCK-axiom

$(-_{12})$: $z - y. - .z - x \leq x - y$.

The weak BCK-algebras indeed form a wider class of algebras.

Example 1. The five-element poset with two maximal chains $0 < a < 1$ and $0 < b < c < 1$, i.e., the non-distributive lattice $\mathbf{N}_5$, provides an example of a wBCK-algebra with the following operation table for $-$:

$-$	0	a	b	c	1
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	0	0
c	c	c	b	0	0
1	1	a	c	b	0

(It is worth to note that the axiom $(-_1)$ together with $(-_9)$ and the following consequence of $(-_7)$:

$$x - y = x \text{ whenever } x \text{ is an atom and } x \not\leq y \tag{2.2}$$

completely determine all but four entries in the table; they likewise help in the further examples. See also the more general condition (2.4).)

Let as denote by $\mathbf{N}_5^1$ the obtained wBCK-lattice. Now, the values $x = a$, $y = c$, $z = 1$ falsify the axiom $(-_{12})$; so $\mathbf{N}_5^1$ is not a BCK-algebra.

It is known well that the class of all BCK algebras is not a variety [37]. As observed in [23], since it consists of those wBCK-algebras satisfying the inequality $(-_{12})$, which can be rewritten as an equation, it follows that wBCK-algebras also do not form a variety. The powerful axiom $(-_{12})$ can be split into a pair of weaker ones.

Theorem 2.4. *A wBCK-algebra is a BCK-algebra if and only if it satisfies the conditions*

$(-_{13})$: *if $x \leq y$, then $x - z \leq y - z$,*

$(-_{14})$: $x - y. - z = x - z. - y$.

Proof. Both these conditions are BCK-theorems—see (3) and (7) in [29]. Conversely, $(-_4)$ and $(-_{13})$ imply that $x - .x - y\colon - z \leq y - z$, wherefrom $(-_{12})$ follows by $(-_{14})$. □

2.2. **Quasi-BCK-algebras.** By a *pocrig* (in full, a partially ordered commutative residuated integral groupoid) [25], we mean an algebra $(A, +, -, 0)$, where $(A, +)$ is a partially ordered commutative groupoid, 0 is its least and simultaneously its neutral element, and $-$ is a binary operation on A characterized by the condition

$$x \leq y + z \text{ if and only if } x - y \leq z.$$

Associative pocrigs (i.e., monoids) are known as pocrims; BCK-algebras are just $(-, 0)$-subreducts of pocrims (see, e.g., [5, Section 2] and references therein). The following analogue of this classical result was proved in the dual form in [25] (Theorem 2).

Proposition 2.5. *An algebra $(A, -, 0)$ is a subreduct of a pocrig if and only if it is a wBCK-algebra which satisfies the isotonicity law $(-_{13})$.*

Following [25], we call a weak BCK-algebra satisfying $(-_{13})$ a *quasi-BCK-algebra*, or just *qBCK-algebra*. The class of all such algebras also is not a variety.

2.3. **Meets in wBCK-algebras.** If, in some wBCK-algebra, elements x and y have the meet $x \wedge y$, then

$$x - y = x - .x \wedge y\,. \tag{2.3}$$

Indeed, $x - .x - y \leq x \wedge y$ by $(-_8)$ and $(-_3)$; then $(-_2)$ gives us the inequality $x - .x \wedge y \leq x - y$. On the other hand, $x - y \leq x - .x \wedge y$ by $(-_4)$. The following useful strengthened version of (2.2):

$$x - y = x \text{ whenever } x \wedge y = 0 \tag{2.4}$$

is an immediate consequence of (2.3).)

Even if the meet of x and y does not exist, the element $x - y$ can be presented in a form $x - z$ with $z \leq x, y$: see Proposition 2.1. Moreover, the following generalization of (2.3) holds in every wBCK-algebra due to $(-_3)$–$(-_5)$ and $(-_8)$:

$$x - y = \min(x - z\colon\ z \leq x, y). \tag{2.5}$$

We may conclude that every wBCK-algebra is completely determined by the structure of its initial segments. See also subsection 6.1.

Most of wBCK-algebras we shall deal with in the further sections will be meet semilattices. By a *wBCK$^\wedge$-algebra* we mean an algebra $(A, \wedge, -, 0)$, where $(A, \wedge)$ is a meet semilattice and $(A, -, 0)$ is a wBCK-algebra w.r.t. its order $\leq$. It follows from Theorem 4 of [23] that the class of all such algebras is equationally definable by semilattice axioms, $(-_3)$, $(-_9)$ and identities

$$x \wedge y - x = 0 \quad\text{and}\quad z - y \leq z - .x \wedge y\ \ .$$

Evidently, the first of these two conditions can be eliminated in favor of $(-_6)$. Again, $(-_6)$ and $(-_9)$ may be replaced by $(-_{11})$.

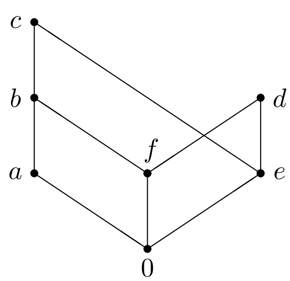
2.4. **Around positive implicative wBCK-algebras.** We adapt a term used for BCK-algebras [28, 29] and call a wBCK-algebra *positive implicative* if it satisfies the *contraction law*

$(-_{15})$: $x - y. - y = x - y.$

A weaker form of the law is the *contraction rule*

$(-_{16})$: if $x - y \leq y$, then $x \leq y$.

Example 2. Let us consider the poset depicted below at the left:



$-$	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	a	0	0	0	a	a	a
b	b	f	0	0	a	b	a
c	c	c	e	0	b	c	c
d	d	d	d	d	0	d	d
e	e	e	e	0	0	0	e
f	f	f	0	0	0	f	0

When equipped with an operation $-$ with the presented table, it becomes a wBCK-algebra in which $(-_{16})$ holds. However, the algebra is not positive implicative: $c - d \neq c - d. - d$.

The right distributive law

$(-_{17})$: $x - z. - .y - z = x - y. - z,$

which implies $(-_{15})$, and is equivalent to it in BCK-algebras [29, Theorem 8], is too strong in the context of wBCK-algebras—it follows from Corollary 2.3 in [14] that a wBCK-algebra satisfying this law is necessarily a BCK-algebra. A direct proof: by $(-_7)$, $(-_4)$ and $(-_{17})$, $z - y. - x \leq z - y. - .x - y = z - x. - y$; this yields $(-_{14})$, and if $x \leq y$, then by $(-_1)$, $(-_{10})$ and $(-_{17})$, $0 = x - y = x - y. - z = x - z. - .y - z$; this yields $(-_{13})$.

In contrary, the *Pierce law*

$(-_{18})$: $x - .y - x = x,$

which was used in [29] as the defining condition for implicative BCK-algebras, turns out to be too weak in the context of wBCK-algebras; we introduce implicative wBCK-algebras in another way in Section 5. Nevertheless, a wBCK-algebra satisfying $(-_{18})$ is always positive implicative: applying this identity twice, we get $x - y. - y = x - y. - :y - .x - y = x - y$. Therefore, $(-_{16})$ also holds in such an algebra. However, the three-element discrete wBCK-chain is an example of a positive implicative wBCK-algebra where $(-_{18})$ fails. We shall see in Section 4 that wBCK-algebras satisfying $(-_{18})$ form an equational class.

Theorem 2.6. *A wBCK-algebra satisfies the Pierce law if and only if the following condition is fulfilled in it:*

$$x \wedge (y - x) = 0. \tag{2.6}$$

Proof. Sufficiency of the condition is evident by (2.3). Now assume that $(-_{18})$ is fulfilled in some wBCK-algebra A, and choose an arbitrary lower bound u of x and $y - x$. Then $u \leq x = x - .y - x \leq x - u$, wherefrom $0 = u - .x - u = u$, i.e., 0 is the single lower bound of x and $y - x$. □

3. Commutative wBCK-algebras

3.1. **Preliminaries.** We extend to weak BCK-algebras also the standard definition of a commutative BCK-algebra [28, 29].

Definition 3.1. A wBCK-algebras is said to be *commutative* if the identity

$(-_{19})$: $x - .x - y = y - .y - x$

holds in it.

Theorem 3.2. *A wBCK-algebra is commutative if and only if it fulfils any of the following (equivalent) conditions:*

$(-_{20})$: *if* $x \leq y$, *then* $x \leq y - .y - x$,
$(-_{21})$: $x \leq y$ *if and only if* $y - .y - x = x$,
$(-_{22})$: $x \leq y$ *if and only if* $x = y - z$ *for some* z.

Proof. Evidently, $(-_{19})$ implies $(-_{20})$ in virtue of $(-_1)$ and $(-_9)$. Conversely, $x - .x - y \leq y - .y - (x - .x - y) \leq y - .y - x$ by $(-_3)$, $(-_{20})$ and $(-_8)$, $(-_4)$.

The condition $(-_{20})$ is included in $(-_{21})$, but is equivalent to the latter by virtue of $(-_3)$ and $(-_7)$. Further, the "if" parts in both $(-_{21})$ and $(-_{22})$ are evident in virtue of $(-_7)$, while the "only if" part of $(-_{22})$ follows from that of $(-_{21})$. At last, to derive the "only if" part of $(-_{21})$ from $(-_{22})$, suppose that $x \leq y$. Then $x = y - z$ for some z, and $y - .y - x = y - :y - .y - z = y - z = x$ due to $(-_5)$. □

The relationship $(-_{21})$ is much more convenient than $(-_{19})$ for checking if a wBCK-algebra is commutative: it does not require looking over all pairs of elements. Thus, the wBCK-algebra $\mathbf{N}_5^1$ from Example 1 satisfies $(-_{21})$. If we change the last line in its operation table by putting $1 - a = b$, $1 - b = a$, $1 - c = a$, we obtain an non-commutative wBCK-algebra $\mathbf{N}_5^2$, in which $1 - .1 - c \neq c$.

We now can extend to commutative wBCK-algebras a result known well for BCK-algebras (see [29, Theorem 3], where it was proved using $(-_{14})$ essentially).

Corollary 3.3. *The operation* $\wedge$ *defined on a commutative wBCK algebra* A *by*

$$x \wedge y := x - .x - y$$

is meet in this poset.

Proof. By (2.3) and $(-_{21})$. □

By a different approach, which relies on the order structure of commutative wBCK-algebras, this result is obtained in subsection 6.1 of [26]. (Similar dual constructions in a context not related to wBCK*-algebras are presented also in [12], [16], [20] and [21].)

The algebra $\mathbf{N}_5^2$ described above shows that a wBCK-semilattice is not necessarily commutative.

3.2. **Equational axioms.** So, every commutative wBCK-algebra can be turned into a wBCK^-algebra, with the meet operation term-definable in a standard way. This implies that the class of commutative wBCK-algebras is equationally definable (subsection 2.3). As shown in [36], the class of commutative BCK-algebras is characterized by equations $(-_6)$, $(-_9)$, $(-_{19})$ and $(-_{14})$. An equational axiom system appropriate for commutative wBCK-algebras can be obtained by replacing the latter identity with a weaker one:

$(-_{23})$: $z - y. - (z - :y - .y - x) = 0$,

which is a version of $(-_4)$.

Theorem 3.4. *An algebra* $(A, -, 0)$ *is a commutative wBCK-algebra with respect to the relation* $\leq$ *defined by* $(-_1)$ *if and only if it satisfies the equations* $(-_6)$, $(-_9)$, $(-_{19})$ *and* $(-_{23})$.

Proof. If A is indeed a commutative wBCK-algebra, then $(-_{23})$ holds in it by virtue of $(-_8)$ and $(-_4)$. Now assume that A satisfies the four equations. With Corollary 2.3 in mind, we shall demonstrate (2.1), $(-_{10})$, $(-_3)$ and $(-_4)$. The relation $\leq$ defined by $(-_1)$ is antisymmetric: if $x \leq y$ and $y \leq x$, then $x = x - 0 = x - .x - y = y - .y - x = y - 0 = y$. The following particular case of $(-_{23})$:

$$x - x. - (x - {:}x - .x - 0) = 0$$

leads us, in virtue of $(-_9)$ and $(-_6)$, to the equation $0 - x = 0$, i.e., to $(-_{10})$. Another particular case of $(-_{23})$,

$$x - .x - y{:} - (x - (x - y. - {:}x - y. - 0)) = 0,$$

similarly reduces to the equation $x - .x - y{:} - x = 0$, which gives us $(-_3)$ in virtue of $(-_{19})$. At last, if $x \leq y$, then $x - y = 0$, $y - .y - x = x - .x - y = x$ and $z - y. - .z - x = z - y. - (z - {:}x - .x - y) = z - y. - (z - {:}y - .y - x) = 0$ (see $(-_{23})$), i.e., $z - y \leq z - x$. We have obtained $(-_4)$. It remains to apply Corollary 2.3. □

With some inessential changes, a dual set of axioms appears in other connection in Section 6.4 of [16]. Equivalent versions of it are used in [12], [20] and [21] to define respectively the classes of Abbot groupoids, implication basic algebras and so called strong I-algebras (see also [10]). Therefore, each of these classes may be identified with that of commutative wBCK*-algebras, and various results obtained in the mentioned papers can be transferred to commutative wBCK-algebras.

In fact, the variety of commutative wBCK-algebras is even 3-based.

Theorem 3.5. *The class of commutative wBCK-algebras is characterized by axioms* $(-_{19})$, $(-_{23})$ *and either* $(-_{11})$ *or*

$(-_{24})$: $x - {:}x - y. - x = x$.

Proof. Evidently, $(-_{11})$ holds in all wBCK-algebras. The identity $(-_{24})$ also holds in every wBCK-algebra: due to $(-_7)$, $x - {:}x - y. - x = x - 0 = x$. Now assume that an algebra $(A, -, 0)$ satisfies any of the two triples of axioms; we should to demonstrate that they imply $(-_6)$ and $(-_9)$ (see Theorem 3.4).

Substituting $0 - y$ for both x and y in $(-_{23})$ and using the particular case $0 - y. - .0 - y = 0 - y$ of $(-_{11})$, we obtain the identity $z - .0 - y{:} - {:}z - .0 - y = 0$. By $(-_{11})$, then $z - z = 0$, which is $(-_6)$. In particular, $0 - y. - .0 - y = 0$; this identity together with the mentioned particular case of $(-_{11})$ provides $(-_9)$.

On the other hand, it is proved in [35] that the following identity is derivable from $(-_{19})$ and $(-_{24})$:

$$x - y. - x = x - x = y - y \ .$$

Then $y - .y - y = y - {:}y - y. - y = y$; so, $(-_{23})$ implies that $z - y. - .z - y = 0$. This gives us $(-_6)$. Further, $x - 0 = x - .x - x = x - {:}x - x. - x = x$, i.e., $(-_9)$ also holds. □

3.3. **Commutative qBCK-algebras.** The next proposition implies that the class of commutative qBCK algebras is a subvariety of the variety of commutative wBCK-algebras.

Proposition 3.6. *A commutative wBCK-algebra is a qBCK-algebra if and only if it satisfies any of the (equivalent) conditions*

$(-_{25})$: $y - .y - x\colon - z \leq y - z$,
$(-_{26})$: $y - u. - z \leq y - z$.

Proof. By $(-_{21})$, $(-_{22})$ and $(-_{7})$. □

Example 3. This subvariety is proper. The bounded lattice $\mathbf{OM}_6$ with six elements $0 < a, b, c, d < 1$ and the operation table

$-$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	a	0	a	a	a	0
b	b	b	0	b	b	0
c	c	c	c	0	c	0
d	d	d	d	d	0	0
1	1	b	a	d	c	0

for an operation $-$ is a commutative wBCK-algebra (in fact, the order dual of the orthoimplication algebra discussed in [3, Remark]). However, it does not satisfy the isotonicity law $(-_{13})$: here, $b \leq 1$, $b - c = b$ and $1 - c = 0$.

Theorem 3.7. *In a commutative qBCK-algebra, if $x \vee y$ exists, then*

$$x \vee y. - x \leq y. \tag{3.1}$$

Proof. Assume that a is a qBCK-algebra, and suppose that $p := x \vee y$ exists in A for some x and y. Since $p - x \leq p$, we have that $p - x. - .p - y \leq p - .p - y \leq y$. By (2.3), then $p - x. - \colon p - x. \wedge .p - y \leq y$. But $p - x. \wedge .p - y \leq p - x, p - y$, whence it follows that $x, y \leq p - (p - x. \wedge .p - y)$ (by $(-_4)$ and $(-_{21})$) and, further, $p \leq p - (p - x. \wedge .p - y)$. Thus $(p - x. \wedge .p - y) = p - .p - (p - x. \wedge .p - y) = 0$, and eventually $p - x \leq y$. □

3.4. **Uniformity.** Let us consider a collection of equivalent conditions on commutative wBCK-algebras.

Lemma 3.8. *The following assertions are equivalent in any commutative wBCK-algebra:*

$(-_{27})$: *if* $x \leq p \leq q$, *then* $p - x = p - \colon p - .q - x$,
$(-_{28})$: *if* $x \leq p \leq q$, *then* $p - .q - x = x$,
$(-_{29})$: *if* $z \leq y$, *then* $x - z. - y = x - y$,
$(-_{30})$: $x - .y - z\colon - y = x - y$.

Proof. Let A be a commutative wBCK-algebra.

(i) It is easily seen that $(-_{27})$ is equivalent to $(-_{28})$. Suppose that $x \leq p \leq q$. The equation $p - x = p - \colon p - .q - x$ implies that $p - .p - x = p - (p - \colon p - .q - x)$, i.e., $p - .q - x = x$: see $(-_5)$ and $(-_{20})$. Clearly, the converse implication also holds.

(ii) Further, $(-_{29})$ is equivalent to $(-_{28})$. Suppose that $z \leq y$. By $(-_4)$ and $(-_7)$, $x - y \leq x - z \leq x$. Then $(-_{28})$ implies that $x - z. - \colon x - .x - y = x - y$. But $x - z. - y \leq x - z. - \colon x - .x - y$ by $(-_3)$ and $(-_4)$. On the other hand, $x - z. \wedge y \leq x \wedge y$, i.e., $x - z. - \colon x - z. - y \leq x - .x - y$, and then $x - z. - \colon x - .x - y \leq x - z. - y$—

see $(-_2)$. Eventually, $x - z. - y = x - y$. Now suppose that $x \le p \le q$. Then $q - p \le q - x$ by $(-_4)$, and then $(-_{29})$ and $(-_{21})$ imply that $x = q - .q - x = q - .q - p\colon - .q - x = p - .q - x$.

(iii) In virtue of $(-_7)$, $(-_{30})$ is a particular case of $(-_{29})$ with $z := y - z$. Conversely, suppose that $z \le y$. Applying $(-_{21})$ and $(-_{30})$, then $(x - z) - y = (x - :y - .y - z) - y = x - y$. □

Observe that $(-_{27})$ can be rewritten as

$$\text{if } x \le p \le q, \text{ then } p - x = p \wedge .q - x\ . \tag{3.2}$$

Let us call a commutative wBCK-algebra *uniform* if it satisfies any of the conditions listed in the lemma. For instance, the wBCK-algebra $\mathbf{OM}_6$ from Example 3 is uniform. Due to $(-_{30})$, the class of uniform commutative wBCK-algebras also is a variety; we shall return to it in Section 5.

4. Orthoimplicative wBCK-algebras

Theorem 10 in [29] says that a BCK-algebra satisfying $(-_{18})$ is both commutative and positively implicative; the converse also holds. Though the BCK-axiom $(-_{12})$ is used in the proofs of these results, they remain valid also in wBCK-algebras.

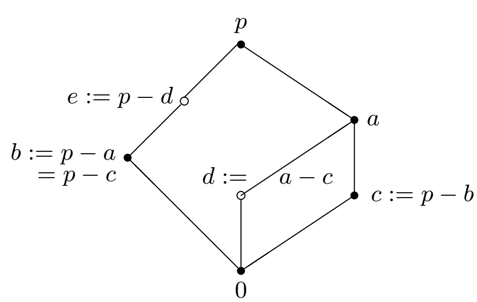
Lemma 4.1. *The Pierce law* $(-_{18})$*, the contraction law* $(-_{15})$ *and the contraction rule* $(-_{16})$ *are equivalent in commutative wBCK-algebras.*

Proof. We already observed in Section 2 that $(-_{18})$ implies $(-_{15})$ and that $(-_{15})$ implies $(-_{16})$. It remains to show that $(-_{18})$ follows from $(-_{16})$. Actually, we shall use Theorem 2.6.

So, assume $(-_{16})$, and suppose that z is a lower bound of x and $y - x$ (hence, $z \le y$). Then $(-_4)$ implies that $y - .y - z \le y - .y - x \le y - z$, whence $y \le y - z$ by and $(-_{16})$. Therefore, $z = y - .y - z = 0$, and 0 is the single lower bound of x and $y - x$, as needed. □

Theorem 4.2. *A wBCK-algebra that satisfies the Pierce law is commutative.*

Proof. Suppose that there is a counterexample—a wBCK-algebra A, in which (2.6) holds, but $(-_{21})$ fails to be true. Then there are elements p and a such that $a \le p$, and $p - .p - a < a$. For convenience, let $b := p - a$ and $c := p - b$; then $c < a$, $p - c = c - a = b$, $a \wedge b = 0 = b \wedge c$ and $0 < a, b, c < p$. See the black dots in the diagram



Let, further, $d := a - c$; so, $c \wedge d = 0$ and $0 < d < a$. This implies that, for $e := p - d$, likewise $d \wedge e = 0$ and $b < e < p$ (as $p - a \le p - d$). For $f := p - e$, further $e \wedge f = 0$, $0 < f \le c$ (as $p - e \le p - b$) and $f = p - .p - d \le d$. This contradicts to the above equality $c \wedge d = 0$. □

BCK-algebras satisfying the Pierce law are commonly called implicative; however, we reserve this term for more special wBCK-algebras (see the next section).

Definition 4.3. A wBCK-algebra satisfying $(-_{18})$ is said to be *orthoimplicative.*

The algebra $\mathbf{OM}_6$ from the Example 3 is an instance of an orthoimplicative wBCK-algebra that is not a qBCK-algebra (see [3]).

Theorem 4.4. *Every orthoimplicative qBCK-algebra is a BCK-algebra.*

Proof. By virtue of Theorem 2.4, it suffices to prove that the exchange law $(-_{14})$ is fulfilled in any orthoimplicative qBCK-algebra A.

Since A satisfies the Pierce law, we conclude from (2.6) that $x - y. - z\colon \wedge\, y \leq x - y. \wedge y = 0$. Since A is also commutative, further $(x-y.-z) -.(x-y.-z)-y = 0$, i.e., $x - y. - z \leq x - y. - z\colon - y$. On the other hand, $(-_7)$ together with $(-_{13})$ imply that $x - y. - z\colon - y \leq x - z. - y$. Thus, $x - y. - z \leq x - z. - y$. The reverse inequality follows by symmetry. □

Since commutative wBCK-algebras form a variety, so do also orthoimplicative wBCK-algebras. Notice that $(-_{24})$ is a particular case of the Pierce law $(-_{18})$. Due to Theorem 3.5, this observation leads to an economical axiom system for orthoimplicative wBCK-algebras.

Proposition 4.5. *An algebra* $(A, -, 0)$ *is an orthoimplicative wBCK-algebra w.r.t. to the relation* $\leq$ *defined by* $(-_1)$ *if and only if it satisfies* $(-_{18})$, $(-_{19})$ *and* $(-_{23})$.

These axioms, together with the identity $(-_6)$ derivable from them, form a system which, up to minor unessential changes, is dual to the system of axioms for orthoimplication algebras of [7] (these differ from orthoimplication algebras of [3]). Ortho-algebras in the sense of [9] is the same class of algebras. Implication orthoalgebras discussed in [13] have some additional axioms, which are in fact redundant. So, all these classes of algebras may be identified with that of orthoimplicative wBCK*-algebras.

5. Implicative wBCK-algebras

The contraction law $(-_{15})$ is a particular case (with $z = y$) of $(-_{29})$. We thus may consider, in the context of commutative wBCK-algebras, uniformity as a strengthening of the property "being positive implicative". Lemma 4.1 then implies that a commutative and uniform wBCK-algebra is orthoimplicative. Notice also that $(-_{21})$ is a particular case of $(-_{28})$; so, a weak BCK algebra is commutative and uniform if and only if it satisfies $(-_{28})$. These observations give rise to the following definition.

Definition 5.1. A wBCK-algebra is said to be *implicative* if it satisfies $(-_{28})$, i.e., is commutative and uniform.

The next theorem presents several conditions that are necessary and sufficient for an orthoimplicative wBCK-algebra to be implicative; each of them is "a half" of a condition from Lemma 3.8. Notice that $(-_{31})$ is subsumed also under the isotonicity law $(-_{13})$.

Theorem 5.2. *A wBCK-algebra is implicative if and only if it is orthoimplicative and satisfies any of the (equivalent) conditions*

$(-_{31})$: *if $x \leq p \leq q$, then $p - x \leq q - x$,*
$(-_{32})$: *if $x \leq p \leq q$, then $p - .q - x \leq x$,*
$(-_{33})$: *if $z \leq y$, then $x - z. - y \leq x - y$,*
$(-_{34})$: $x - .y - z{:} - y \leq x - y$.

Proof. Evidently, $(-_{31})$ and $(-_{32})$ are equivalent by $(-_2)$. Further, an inspection of items (ii) and (iii) in the proof of Lemma 3.8 shows that the equivalence of $(-_{32})$–$(-_{34})$ can be proved just in the same way (of course, omitting the unnecessary portions of the arguments).

We have already noticed that an implicative wBCK-algebra always is orthoimplicative. This remark ends the proof of the “only if part” of the theorem. Its ”if” part is proved at the end of the subsequent section: see Proposition 6.8. □

Orthoimplicative wBCK-algebras form a proper subclass of implicative wBCK-algebras.

Example 4. Let A be a bounded lattice with five atoms a, b, c, d, e and five coatoms f, g, h, i, j such that $f = e \vee d$, $g = c \vee e$, $h = b \vee d$, $i = a \vee c$, $j = a \vee b$, all other joins of incomparable pairs of elements being equal to 1. (This is the self-dual lattice presented in [13, Example].) The operation $-$ with the table

$-$	0	a	b	c	d	e	f	g	h	i	j	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	a	0	a	a	a	a	a	a	a	0	0	0
b	b	b	0	b	b	b	b	b	0	b	0	0
c	c	c	c	0	c	c	c	0	c	0	c	0
d	d	d	d	d	0	d	0	d	0	d	d	0
e	e	e	e	e	e	0	0	0	e	e	e	0
f	f	f	f	f	e	d	0	d	e	f	f	0
g	g	g	g	e	g	c	c	0	g	e	g	0
h	h	h	d	h	b	h	b	h	0	h	d	0
i	i	c	i	a	i	i	i	a	i	0	c	0
j	j	b	a	j	j	j	j	j	a	b	0	0
1	1	f	g	h	i	j	a	b	c	d	e	0

turns it into an orthoimplicative algebra (cf. Corollary 6.4(a)) in which $(-_{31})$ fails: $a \leq j \leq 1$, but $j - a \not\leq 1 - a$.

Theorem 5.3. *An algebra $(A, -, 0)$ is an implicative wBCK-algebra w.r.t. to the relation $\leq$ defined by $(-_1)$ if and only if it satisfies $(-_6)$, $(-_{18})$, $(-_{19})$ and $(-_{30})$.*

Proof. Of course, the four listed identities are fulfilled in an implicative algebra. On the other hand, the order dual of an algebra $(A, -, 0)$ satisfying these identities is essentially an orthoimplication algebra in the sense of [3] (and conversely). By Lemma 1 of that paper, such an algebra A satisfies also $(-_{10})$, $(-_4)$ and (2.1). It satisfies also $(-_8)$ (put $y = x$ in $(-_{30})$) and, hence, $(-_3)$. So, A is a wBCK-algebra (Corollary 2.3), which is implicative by definition (see Lemma 3.8). □

Therefore, orthoimplication algebras may be identified with implicative wBCK*-algebras.

6. Some structure theorems for commutative wBCK-algebras

6.1. **Preliminaries.** Theorem 2.3 in [24] discovers the structure of initial segments of wBCK-algebras. In particular, every initial segment of a wBCK-algebra is its subalgebra. Proposition 6.1 below is a slightly improved version of the theorem.

A unary operation $^+$ on a bounded poset is called a *dual Galois complementation* (or just *g*-complementation*) [24] if it satisfies the conditions

$$x^{++} \leq x, \quad \text{if } x \leq y, \text{ then } y^+ \leq x^+, \quad 0^+ = 1$$

(then $x^+ = 0$ iff $x = 1$). A poset with 0 is said to be *sectionally g*-complemented*, if every section $[0,p]$ in it is g*-complemented. Observe that every bounded poset admits the *discrete g*-complementation* defined by

$$x^+ = 0 \text{ if } x = 1, \text{ and } x^+ = 1 \text{ otherwise.}$$

Proposition 6.1. *Let A be a poset with the least element 0, a binary operation $-$ and, for every $p \in A$, a unary operation ^+_p on $[0,p]$. The following assertions are equivalent:*

(a) $(A, -, 0)$ *is a weak BCK-algebra, and $x^+_p = p - x$ for every $p \in A$ and all $x \leq p$.*

(b) *Every operation ^+_p is a g*-complementation, and*

$$x - y = \min\{z^+_x \colon\ z \leq x, y\} \text{ for all } x, y \in A.$$

If the poset A is a meet semilattice, then the latter condition reduces to

$$x - y = (x \wedge y)^+_x \text{ for all } x, y \in A.$$

When speaking on a wBCK-algebra as sectionally g*-complemented, or on a sectionally g*-complemented poset as a wBCK-algebra, we shall have in mind just the operations ^+_p defined in (a) and the subtraction described in (b). The equalities (2.3) and (2.5) are easy consequences of this Proposition. Notice that there is a bijective correspondence between semilattice ordered wBCK-algebras and sectionally g*-complemented semilattices. (This correspondence is functorial.) Generally, the transfer from wBCK-algebras to sectionally g*-complemented posets is injective, and, of course, different g*-complemented posets cannot support the same wBCK-algebra. Every sectionally g*-complemented poset with discrete initial segments supports a discrete wBCK-algebra (and conversely). For all that, not every sectionally g*-complemented poset gives raise to a wBCK-algebra.

Example 5. The five-elements poset consisting of four maximal chains $0 < a < c$, $0 < b < c$, $0 < a < d$ and $0 < b < d$ may be regarded as sectionally g*-complemented with $a^+_c = a^+_d = b$ and $b^+_c = b^+_d = a$. This allows us to define $x - y$ uniquely for all values of x and y except for $x = c, y = d$ and $x = d, y = c$. The reason for the exceptions is that neither a^+_c and b^+_c nor a^+_d and b^+_d are comparable.

To proceed, we introduce several particular types of g*-complementation; some of them were discussed in [24]. A unary operation $^+$ on a bounded poset is called

- a *semicomplementation* (*s-complementation*) if it satisfies the conditions

  $x \wedge x^+$ exists and equals to 0, and $x^+ = 0$ only if $x = 1$,

- a *dual Brouwerian complementation* (*b*-complementation*) if it is a g*-complementation such that $x \vee x^+$ exists and equals to 1 for all x,
- a *De Morgan complementation* (*m-complementation*), if it is an idempotent g*-complementation,

- an *orthocomplementation* (*o-complementation*), if it is an idempotent b*-complementation.

Therefore, an o-complementation may be characterized also as m-complementation which is an s-complementation (recall the note subsequent to the definition of g*-complementation).

Let the symbol @ stand for any of symbols s, b*, m, o. A poset with 0 is said to be *sectionally @-complemented* if every section $[0.p]$ in it is equipped with a @-complementation.

The subsequent proposition is a rewording of [24, Theorem 3.2]. We give it a short independent proof.

Proposition 6.2. *A wBCK-algebra satisfies the contraction rule* $(-_{16})$ *if and only if it is sectionally b*-complemented.*

Proof. Let A be a wBCK-algebra. By Proposition 6.1, it (being sectionally g*-complemented) is sectionally b*-complemented if and only if $p - x. \vee x$ exists and equals to p whenever $x \leq p$.

Assume that $(-_{16})$ is fulfilled in A, and suppose that $x \leq p$. Clearly, then p is an upper bound of x and $p - x$. If z is one more upper bound, then $p - z \leq p - x \leq z$ by $(-_4)$, and, further, $p \leq z$ by $(-_{16})$. Therefore, p is indeed the least upper bound of x and $p - x$. Thus, A is sectionally b*-complemented.

Conversely, assume that A is sectionally b*-complemented. Then $x = x - y. \vee x - .x - y \leq y$ whenever $x - y \leq y$: see $(-_7)$ and $(-_3)$. $(-_8)$, $(-_5)$ and $(-_3)$, and $(-_{16})$ immediately follows. □

6.2. **Commutative and orthoimplicative wBCK-algebras.** We can say more about the structure of commutative wBCK-algebras. An m-complemented lattice is called a *non-distributive De Morgan lattice*. A meet semilattice in which every pair of elements bounded above has the join is known as a *nearlattice*. It follows that every initial section in a nearlattice is a lattice. Lemma 2.5 in [24] says that a sectionally m-complemented poset is a nearlattice if and only if it is a wBCK-algebra.

Theorem 6.3. (a) *A wBCK-algebra is commutative if and only if its sectional g*-complementations are idempotent.*
(b) *A commutative wBCK-algebra is a nearlattice in which every section is a non-distributive De Morgan lattice.*

Proof. (a) Due to Proposition 6.1, the operations $\overset{+}{p}$ in a wBCK-algebra are idempotent, i.e., are m-complementations, just in the case when $(-_{20})$ holds. Then (b) follows immediately from definitions and the mentioned lemma from [24]. □

Similar results (without relating them with wBCK-algebras, of course) were obtained already in [12] and other papers mentioned in Section 3 (p. 6, 7). Corollary 6.4 in [26] is a slightly more general version of item (a) of the above theorem.

It is now well-known that a bounded commutative BCK-algebra can be equipped with the structure of MV-algebra; even more, classes of bounded BCK-algebras and MV-algebras are term-equivalent [22]. In [20], a similar equivalence is stated for bounded implicative basic algebras (i.e., bounded commutative wBCK*-algebras, see Section 3) and a version of non-associative MV-algebras known as basic algebras [16, 17]. Lattices studied in [6, 10] in connection with MV-algebras also are in fact

bounded commutative wBCK*-algebras, while lattices from an earlier paper [15] are essentially even (bounded commutative) BCK*-algebras. The reader could derive from [6, Theorem 2] necessary and sufficient conditions for sections of a bounded commutative wBCK-algebra to be distributive De Morgan lattices.

An o-complemented lattice is also called an *ortholattice*.

Theorem 6.4. (a) *A wBCK-algebra is orthoimplicative if and only if it is sectionally s-complemented.*

(b) *An orthoimplicative wBCK-algebra is a nearlattice in which every section is an ortholattice.*

Proof. (a) Let A be a wBCK-algebra. By Proposition 6.1, it (being sectionally g*-complemented) is sectionally s-complemented if and only if $x \wedge .p - x = 0$ whenever $x \leq p$, i.e., if and only if $p \wedge x. \wedge :p - .p \wedge x = 0$. By $(-_7)$ and (2.3), the latter condition is equivalent to the identity $x \wedge .p - x = 0$, which is (2.6), and, hence, to the Pierce law, as needed.

(b) By Lemma 4.2, an orthoimplicative wBCK-algebra is commutative. Now use the preceding theorem, Proposition 6.2 and Lemma 4.1. ☐

For related dual results, see the papers mentioned on p. 10 in Section 4. The above theorem has the following immediate consequence.

Corollary 6.5. *Every sectionally s-complemented wBCK-algebra is sectionally o-complemented.*

As shown in [31], a sectionally s-complemented poset is distributive if and only if it is 0-distributive in the sense explained in that paper. It follows that an orthoimplicative wBCK-algebra has Boolean sections if and only if it is 0-distributive. Congruence properties of sectionally s-complemented distributive nearlattices have been studied in [34].

6.3. **Implicative wBCK-algebras.** It is known that implication algebras of [1, 2] (known also as Tarski algebras) coincide with implicative BCK*-algebras; see, e.g., [33]. Any implicative BCK-algebra is a nearlattice with Boolean sections. It follows that a bounded BCK-algebra is implicative if and only if it is a Boolean lattice (with the complementation $^+_1$); see [28, 29]. These results have counterparts for weak BCK-algebras, with orthomodular lattices instead of Boolean ones. Recall that an orthomodular lattice is an ortholattice in which $y = x \vee (y \wedge x^+)$ whenever $x \leq y$.

Theorem 6.6. (a) *A wBCK-algebra is implicative if and only if it is sectionally m-complemented and satisfies the condition*

$$\text{if } x \leq p \leq q, \text{ then } x^+_p = p \wedge x^+_q. \tag{6.1}$$

(b) *An implicative wBCK-algebra is a nearlattice in which every section is an orthomodular lattice.*

(c) *A bounded wBCK-algebra is implicative if and only if it is an orthomodular lattice.*

Proof. (a) By Theorems 6.3(a) and 6.1, condition (3.2), and Lemma 3.8.

(b) An implicative wBCK-algebra is, in particular, orthoimplicative. By Theorem 6.4(b), it is a nearlattice with initial sections ortholattices. Now suppose

that $x \leq y \leq q$. Then $y = x \vee x_y^+$, and we, applying (6.1), come to the equality $y = x \vee (y \wedge x_q^+)$. Consequently, the ortholattice $[0, q]$ is orthomodular.

(c) follows from (b), for every orthomodular lattice is sectionally orthomodular and satisfies (6.1). □

In connection with (b), cf. Theorem 4 in [11] or Proposition in [8] and their proofs. Semi-orthomodular lattices of [3] are order duals of those sectionally orthomodular meet semilattices that satisfy (6.1); so, they can be characterized also as implicative wBCK*-algebras (see the connection between wBCK-algebras, nearlattices and sectionally m-complemented posets mentioned just before Theorem 6.3). Further, a generalized orthomodular lattice [30, 4] can be defined as a sectionally orthomodular lattice satisfying (6.1). We conclude that a wBCK-lattice is implicative if and only if it is a generalized orthomodular lattice.

It should be noted that the converse of (b) does not hold true: not every sectionally orthomodular wBCK-algebra satisfies 6.1 and therefore is implicative. Let us call a wBCK-algebra A *semi-implicative* if it is an order dual of an orthomodular implication algebra in the sense of [18, 19]. It follows from Theorems 3–5 of [18] that A is semi-implicative if and only if it is sectionally orthomodular. Therefore, every semi-implicative wBCK-algebra is orthoimplicative, and every implicative wBCK-algebra is semi-implicative. Of course, the orthoimplicative wBCK-algebra from Example 4 is not semi-implicative; cf. [19, Section 4]. In its turn, Theorem 4.2 in [32] shows an orthomodular implication algebra that is not an orthoimplication algebra in the sense of [3]. In terms of the present paper (cf. the note at the end of Section 5), it is a semi-implicative wBCK*-algebra that is not implicative .

In some situations, however, the requirement (6.1) in Theorem 6.6(a) can be weakened.

Theorem 6.7. *A wBCK-algebra is implicative if and only if it is sectionally s-complemented and satisfies the condition*

$$\text{if } x \leq p \leq q, \text{ then } x_p^+ \leq x_q^+, \tag{6.2}$$

Proof. Let A be some wBCK-algebra. If it is implicative, then, being orthoimplicative, it is sectionally s-complemented by Theorem 6.4(a). Moreover, $(-_{31})$ implies (6.2).

Now assume that A is sectionally s-complemented and satisfies (6.2). Then it is sectionally o-complemented nearlattice (Corollary 6.5), and we may use De Morgan duality laws in every section. Suppose that $x \leq p \leq q$. Then $x_p^+ \leq p$ and $x = (x_p^+)_p^+ \leq (x_p^+)_q^+$. Put $z := (x_p^+)_q^+$; clearly, $x \leq z \leq q$. As $x_z^+ \leq z$ and $x_z^+ \leq x_q^+$, further $z = x \vee x_z^+ \leq x \vee .z \wedge x_q^+ \leq z$, wherefrom $z = x \vee .z \wedge x_q^+$. Now $x_p^+ = z_q^+ = x_q^+ \wedge .z_q^+ \vee x = x_q^+ \wedge .x_p^+ \vee x = x_q^+ \wedge p$, i.e., (6.1) also is fulfilled. By the previous theorem, A is implicative. □

Sectionally orthocomplemented posets satisfying (6.2) and admitting joins of certain pairs of elements are studied in Section 5 of [26] under the name 'relatively orthocomplemented posets'.

By Proposition 6.1, the conditions (6.2) and $(-_{31})$ are equivalent. Then, in view of Theorem 6.4, the last theorem implies, in particular, the following result, which completes the proof of Theorem 5.2.

Proposition 6.8. *An orthoimplicative wBCK-algebra satisfying $(-_{27})$ is implicative.*

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